# Return to the Riemann Integral

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To J. T. Schwartz, on his 65th birthday

**§1. INTRODUCTION.** It is well known that the Riemann integral is not adequate for advanced mathematics, since there are many functions that are not Riemann-integrable, and since the integral does not possess sufficiently strong convergence theorems. To correct these deficiencies, Lebesgue developed his integral around the turn of the present century, and his integral has become the "official" integral in mathematical research.

However, there are also difficulties with the Lebesgue integral:

(1) There exist functions $F$ that are differentiable at every point, but such that their derivatives $F'$ are not Lebesgue integrable. Thus an added hypothesis is necessary to validate the formula

$$\int_a^b F' = F(b) - F(a). \tag{1a}$$

As one consequence, theorems justifying the substitution formula

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b (f \circ \varphi)\, \varphi' \tag{1b}$$

become unnecessarily complicated.

(2) Some improper integrals, such as the important Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x}\, dx, \tag{1c}$$

do not exist as Lebesgue integrals (since $|x^{-1} \sin x|$ is not Lebesgue integrable).

(3) A considerable amount of measure theory needs to be developed before the Lebesgue integral can be defined.

It is the position of the present author that *the time has come to discard the Lebesgue integral as the primary integral.* We should replace it with a general form of the Riemann integral that—surprisingly enough—is *more* general than the Lebesgue integral and corrects the above difficulties. This generalization was discovered by Jaroslav Kurzweil and Ralph Henstock around 1960, but for some reason it has not become well known. Its definition is "Riemann-like", but its power is "super-Lebesgue". It is our view that we should not try to teach proofs to beginning calculus students, but that we should equip them with theorems to apply. Somewhat later, serious undergraduate students should be expected to understand appropriate proofs. We believe that most American undergraduates are not ready to study the Lebesgue integral, but that they are capable of mastering a (somewhat stripped-down) version of the generalized Riemann integral. In §§2–10, we will provide an outline of such a version, with some side remarks to those who already know about the Lebesgue integral.

**Historical remark.** Over 20 years ago, E. J. McShane **[7]** made an eloquent argument for replacing the usual measure-theoretic approach to the Lebesgue integral by a Riemann-type approach that is afforded by the generalized Riemann integral. He later published a book **[8]** that could be used as a text for undergraduates in such a course. It is the present author's opinion that McShane was (i) overly optimistic in believing that the full Lebesgue integral can be taught to undergraduates, and (ii) overly conservative in developing the Lebesgue integral and not the generalized Riemann integral, which is more powerful and, we believe, conceptually simpler.

**§2. BASIC DEFINITIONS.** For the sake of simplicity, we will limit most of our remarks to the case of an interval $I := [a, b]$, $a < b$, in $\boldsymbol{R}$ and functions with (finite!) values in $\boldsymbol{R}$.

A *partition* of $I$ is a finite collection of non-overlapping nondegenerate closed intervals $\{I_i\}_{i=1}^n$ whose union is $I$. Usually the partition is ordered and the intervals are specified by their end points; thus $I_i := [x_{i-1}, x_i]$, where

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b. \tag{2a}$$

A *tagged partition* $P := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a finite set of ordered pairs, where the closed intervals $I_i = [x_{i-1}, x_i]$ form a partition of $I$ and the numbers $t_i \in I_i$ are called the corresponding *tags*. If $P := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $I$ and $f\colon I \to \boldsymbol{R}$ is a function, then the *Riemann sum* $S(f; P)$ of $f$ corresponding to $P$ is the number

$$S(f; P) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}). \tag{2b}$$

The usual definition of the Riemann integral can be phrased: The number $A \in \boldsymbol{R}$ is the *Riemann integral* of $f\colon I \to \boldsymbol{R}$ if for every $\varepsilon > 0$ there exists a *constant* $\delta_\varepsilon > 0$ such that if $P := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $I$ satisfying $0 < x_i - x_{i-1} \le \delta_\varepsilon$ for $i = 1, \dots, n$, then

$$|S(f; P) - A| \le \varepsilon. \tag{2c}$$

It turns out that the use of a *constant* $\delta_\varepsilon > 0$ restricts the Riemann integral quite considerably. The generalized Riemann integral is obtained by allowing $\delta_\varepsilon$ to be any strictly positive *function* on $I$. At first glance, that change seems to be very minor, but it turns out to make a profound difference in the properties of the resulting integral.

A strictly positive function $\delta$ on $I$ is called a *gauge* on $I$. If $\delta$ is a gauge on $I$ and $P := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $I$, we say that $P$ is *$\delta$-fine* in case

$$0 < x_i - x_{i-1} \le \delta(t_i) \qquad \text{for} \quad i = 1, \dots, n. \tag{2d}$$

The Nested Intervals Theorem implies that, given any gauge $\delta$ on $I$, there exist $\delta$-fine partitions of $I$. The definition of the generalized Riemann integral differs from that of the ordinary Riemann integral by allowing nonconstant gauges.

**(2.1) Definition.** A number $B \in \boldsymbol{R}$ is the *generalized Riemann integral* of a function $f\colon I \to \boldsymbol{R}$ if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon$ on $I$ such that if $P := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any partition of $I$ that is $\delta_\varepsilon$-fine, then

$$|S(f;P) - B| \le \varepsilon. \tag{2e}$$

In this case we will write $f \in \mathscr{R}^*(I)$ and denote $B = \int_I f = \int_a^b f$.

To show directly that $f \in \mathscr{R}^*(I)$, one must produce a suitable gauge $\delta_\varepsilon$ on $I$ for any given $\varepsilon > 0$. However, there is a Cauchy condition for integrability and it is usually more convenient to use that condition (or other theorems) to establish the integrability of functions. It is an easy exercise to show that if $f \in \mathscr{R}^*(I)$, then the number $B$ in (2e) is uniquely determined. Further, one can change the values of an integrable function on a *null* set without affecting the integrability or the value of the integral. The collection $\mathscr{R}^*(I)$ is a vector space and admits pointwise multiplication by functions of bounded variation. (Recall the Abel and Dirichlet Tests for non-absolutely convergent series.)

**Remark.** In establishing the details of the theory, it is found to be convenient to use a slightly different definition of $\delta$-fineness.

**§3. SOME EXAMPLES.** We now give some examples of functions that belong to the collection $\mathscr{R}^*(I)$.

**(3.1)** Every Riemann integrable function on $I$ is in $\mathscr{R}^*(I)$.

This follows from the fact that the gauge can be a strictly positive constant function. Thus, every continuous function on $I$ is in $\mathscr{R}^*(I)$, and every step function on $I$ is in $\mathscr{R}^*(I)$.

**(3.2)** If $h\colon [0,1] \to \boldsymbol{R}$ is Dirichlet's function (= the characteristic function of the rational numbers in $[0,1]$), then $h \in \mathscr{R}^*([0,1])$ and $\int_0^1 h = 0$.

To prove this assertion, we will define an appropriate gauge $\delta_\varepsilon$. First we enumerate these rational numbers as $\{r_1, r_2, \dots\}$. We define $\delta_\varepsilon(r_i) := \varepsilon/2^{i+1}$, and if $x \in [0,1]$ is irrational we define $\delta_\varepsilon(x) := 1$; clearly $\delta_\varepsilon$ is a gauge on $[0,1]$. If $P$ is a $\delta_\varepsilon$-fine tagged partition, there can be at most two subintervals in $P$ that have the number $r_i$ as tag, and the length of each of those subintervals is $\le \varepsilon/2^{i+1}$. Hence the contribution to $S(h;P)$ from subintervals with tag $r_i$ is $\le \varepsilon/2^i$. Since the terms in $S(h;P)$ with tags at irrational points contribute 0, we readily see that

$$0 \le S(h;P) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $h \in \mathscr{R}^*([0,1])$ and that $\int_0^1 h = 0$.

**(3.3)** Every Lebesgue integrable function on $I$ is in $\mathscr{R}^*(I)$.

A proof of this (non-obvious) result requires an understanding of the Lebesgue integral. Thus the teacher will want to know a proof, but the student is not concerned with this result or its proof.

**(3.4)** There exist functions in $\mathscr{R}^*(I)$ that do not belong to $\mathscr{L}(I)$.

Indeed, the function $F(x) := x^2 \cos(\pi/x^2)$ for $x \in (0,1]$ and $F(0) := 0$ is readily seen to be differentiable at every point of $[0,1]$. It will be seen in §4 that this implies that $f := F' \in \mathscr{R}^*([0,1])$. [However, since $F$ is not absolutely continuous on $[0,1]$ the teacher will understand that $f \notin \mathscr{L}([0,1])$.]

**(3.5)** If $\Sigma_{n=1}^{\infty} a_n$ is any convergent series, then one can define $h(x) := 2^n a_n$ for $x \in (1/2^n, 1/2^{n-1}]$ for $n \in N$ and $h(0) := 0$. A gauge can be constructed to show that $h \in \mathscr{R}^*([0,1])$ and that

$$\int_0^1 h = \sum_{n=1}^{\infty} a_n.$$

Moreover, $|h| \in \mathscr{R}^*([0,1])$ if and only if the series $\Sigma_{n=1}^{\infty} a_n$ is absolutely convergent [if and only if $h \in \mathscr{L}([0,1])$].

Example (3.5) shows that the absolute value of function in $\mathscr{R}^*(I)$ is not necessarily in $\mathscr{R}^*(I)$. Thus the generalized Riemann integral is not an "absolute integral". That is why the Dirichlet integrand in (1c) can be in $\mathscr{R}^*([0,\infty))$.

**§4. THE FUNDAMENTAL THEOREM.** We have noted in §1 that the Lebesgue integral is not powerful enough to integrate *every* derivative. That fact led Denjoy and Perron to develop their (very different) theories of integration. The details and subtleties of these theories of integration are quite considerable (see [2]). This stands in marked contrast with the generalized Riemann integrals, for which, as we will now see, the details are remarkably simple.

**(4.1) Fundamental Theorem.** *If $F\colon [a,b] \to R$ is differentiable at every point of $I := [a,b]$, then $f = F'$ belongs to $\mathscr{R}^*(I)$ and*

$$\int_a^b f = F(b) - F(a). \tag{4a}$$

*Proof:* If $t \in I$, since $f(t) = F'(t)$ exists, given $\varepsilon > 0$ there exists $\delta_\varepsilon(t) > 0$ such that if $0 < |z - t| \le \delta_\varepsilon(t)$, $z \in I$, then

$$\left| \frac{F(z) - F(t)}{z - t} - f(t) \right| \le \varepsilon.$$

Thus a gauge $\delta_\varepsilon$ has been defined on $I$. Further, if $|z - t| \le \delta_\varepsilon(t)$, $z \in I$, then

$$|F(z) - F(t) - (z - t)f(t)| \le \varepsilon |z - t|.$$

Hence, if $a \le u \le t \le v \le b$ and $0 < v - u \le \delta_\varepsilon(t)$, then it follows that

$$\begin{aligned} &|F(v) - F(u) - (v - u)f(t)| \\ &\quad \le |F(v) - F(t) - (v - t)f(t)| + |F(t) - F(u) - (t - u)f(t)| \\ &\quad \le \varepsilon(v - t) + \varepsilon(t - u) = \varepsilon(v - u). \end{aligned}$$

If $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a $\delta_\varepsilon$-fine partition of $I$, then the telescoping sum $F(b) - F(a) = \Sigma_{i=1}^n \{F(x_i) - F(x_{i-1})\}$ satisfies the approximation

$$\begin{aligned} |F(b) - F(a) - S(f;P)| &= \left| \sum_{i=1}^n \{F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\} \right| \\ &\le \sum_{i=1}^n |F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \\ &\le \sum_{i=1}^n \varepsilon(x_i - x_{i-1}) = \varepsilon(b - a). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that $f$ is in $\mathscr{R}^*(I)$ and establishes (4a).

It is not difficult to extend the Fundamental Theorem (4.1) to a function that is the derivative of a continuous function at all but a countable set of points in $I$. Thus, it follows that the function defined by $f(x) := 1/\sqrt{x}$ for $x \in (0,1]$ and $f(0) := 0$ is in $\mathscr{R}^*([0,1])$, since it is the derivative of the function $F(x) := 2\sqrt{x}$ except at $x = 0$. Thus we have

$$\int_0^1 f = \int_0^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\,\Big|_0^1 = 2.$$

Of course, the function $f$ is well known to have an *improper* Riemann integral; we have just seen that it has an ordinary *generalized* Riemann integral.

**§5. SUBSTITUTION THEOREMS.** In view of the simplicity of the Fundamental Theorem (4.1), one can make corresponding improvements in the theorems justifying the familiar "substitution formula":

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b (f \circ \varphi)\,\varphi'. \tag{5a}$$

Although this formula is a basic tool in analysis, it is seldom stated (or proved) with the generality required for nontrivial use. We will content ourself with two theorems that the generalized Riemann integral renders valid.

**(5.1) Substitution Theorem, I.** *Let* $\varphi\colon [a,b] \to \mathbf{R}$ *be differentiable on* $I := [a,b]$ *and let* $F$ *be differentiable on the interval* $\varphi(I)$. *If* $f(x) = F'(x)$ *for all* $x \in \varphi(I)$, *then equation* (5a) *holds.*

*Proof:* It follows from the Chain Rule that $(F \circ \varphi)'(x) = (f \circ \varphi)(x)\varphi'(x)$ for all $x \in I$. Two applications of the Fundamental Theorem (4.1) imply that

$$\int_a^b (f \circ \varphi)\,\varphi' = F \circ \varphi\big|_a^b = F\big|_{\varphi(a)}^{\varphi(b)} = \int_{\varphi(a)}^{\varphi(b)} f.$$

The proof of the next result is more subtle.

**(5.2) Substitution Theorem, II.** *Let* $\varphi$ *be a strictly increasing and differentiable mapping of* $I := [a,b]$ *onto* $\varphi(I) = [\varphi(a), \varphi(b)]$. *Then* $f$ *belongs to* $\mathscr{R}^*(\varphi(I))$ *if and only if* $(f \circ \varphi)\varphi'$ *belongs to* $\mathscr{R}^*(I)$. *In this case* (5a) *holds.*

Both (5.1) and (5.2) can be extended to more general circumstances.

**§6. IMPROPER INTEGRALS.** One of the remarkable properties of the generalized Riemann integral (that is not shared by either the ordinary Riemann integral or the Lebesgue integral) is the following theorem due to H. Hake.

**(6.1) Hake's Theorem.** *A function* $f$ *belongs to* $\mathscr{R}^*([a,b])$ *if and only if it belongs to* $\mathscr{R}^*([a,c])$ *for every* $c \in (a,b)$ *and* $\lim_{c \to b-} \int_a^c f$ *exists in* $\mathbf{R}$. *In this case* $\int_a^b f = \lim_{c \to b-} \int_a^c f$.

One can interpret Hake's Theorem as asserting: *The generalized Riemann integral cannot be extended by adjoining functions with "improper integrals"*. In other words, if the "improper integral" exists, then the integral exists as (an ordinary) generalized Riemann integral.

The student would be interested in the half of the theorem asserting that the integral can be evaluated as a limit; the proof of that part is rather easy. The harder part of the proof is of interest only to the teacher, since only the teacher believes in improper integrals.

**§7. CHARACTERIZATION OF INDEFINITE INTEGRALS.** In §4 we discussed one aspect of the Fundamental Theorem, namely the integrability of any derivative. The other aspect of the Fundamental Theorem pertains to the differentiation of the *indefinite integral of* $f$, which is the function $F$ defined by

$$F(x) := \int_a^x f \qquad \text{for} \quad x \in [a,b]. \tag{7a}$$

In an undergraduate course, it would probably be best to content oneself with showing that $F'(c) = f(c)$ at every point $c \in I$ where $f$ is continuous.

[The teacher, of course, should know more. In fact, using the Vitali Covering Theorem, one can show that $F$ is differentiable almost everywhere and that

$$F'(x) = f(x) \quad \text{almost everywhere}. \tag{7b}$$

However, the proof of this fact is a bit too much for most undergraduates. The teacher should know that there is an extension of Lebesgue's characterization of indefinite integrals that is valid for the generalized Riemann integral. Somewhat imprecisely stated, a function $F$ is an indefinite integral of $f \in \mathscr{R}^*(I)$ if and only if (i) $F'(x) = f(x)$ almost everywhere in $I$, and (ii) on the set where (i) does not hold, then $F$ has "arbitrarily small variation". This characterization can be used to give a proof that $\mathscr{L}(I) \subset \mathscr{R}^*(I)$, of interest to the teacher. It also implies that $f \in \mathscr{L}(I)$ if and only if both $f$ and $|f|$ belong to $\mathscr{R}^*(I)$.]

However, from the student's standpoint, it would be appropriate merely *to define* the space of *Lebesgue integrable functions* to be:

$$\mathscr{L}(I) := \{f \in \mathscr{R}^*(I) : |f| \in \mathscr{R}^*(I)\},$$

and to make $\mathscr{L}(I)$ into a semi-normed space under

$$\|f\|_1 := \int_I |f|.$$

**§8. CONVERGENCE THEOREMS.** One of the main reasons for the interest in the Lebesgue integral is its convergence theorems. It is quite surprising that they also hold in $\mathscr{R}^*(I)$.

It is easy to prove that if $(f_n)$ is a sequence in $\mathscr{R}^*([a,b])$ that converges *uniformly* to $f$ on $[a,b]$, then $f \in \mathscr{R}^*([a,b])$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. \tag{8a}$$

However, the Monotone Convergence Theorem is also true in $\mathscr{R}^*(I)$.

**(8.1) Monotone Convergence Theorem.** *Let* $(f_n)$ *be a sequence in* $\mathscr{R}^*([a,b])$ *that is monotone increasing*:

$$f_1(x) \le \cdots \le f_n(x) \le f_{n+1}(x) \le \cdots \qquad \textit{for all} \quad x \in [a,b],$$

*and let* $f(x) = \lim_n f_n(x) \in \boldsymbol{R}$ *for all* $x \in [a,b]$. *Then* $f \in \mathscr{R}^*([a,b])$ *if and only if*

$$\sup_n \int_a^b f_n < \infty.$$

*In this case,* (8a) *holds.*

From this one can prove Fatou's Lemma and the following version of the Dominated Convergence Theorem.

**(8.2) Dominated Convergence Theorem.** *Let $(f_n)$ be a sequence in $\mathscr{R}^*([a,b])$, let $g, h \in \mathscr{R}^*([a,b])$ be such that*

$$g(x) \leq f_n(x) \leq h(x) \qquad \text{for all} \qquad x \in [a,b],$$

*and let $f(x) = \lim_n f_n(x) \in \boldsymbol{R}$ for all $x \in [a,b]$. Then $f \in \mathscr{R}^*([a,b])$ and* (8a) *holds.*

These proofs do not need any measure theory, but they may be slightly out of the range of most undergraduates.

**§9. MEASURE THEORY.** Ultimately, it is desirable that students learn some measure theory. We now suggest how that theory can be developed from the generalized Riemann integral. (The situation is slightly more complicated for an infinite interval.)

As usual, we define a *null set* in $I := [a,b]$ to be a set that can be covered by a countable union of intervals with arbitrarily small total length. We define a function $f\colon I \to \boldsymbol{R}$ to be *measurable* if there exists a sequence of step (or continuous) functions on $I$ that converges to $f$ almost everywhere (that is, on the complement of a null set). One can now relate this notion to the generalized Riemann integral.

**(9.1) Measurability Theorem.** *Every $f \in \mathscr{R}^*(I)$ is measurable on $I$.*

Indeed, $f$ is equal almost everywhere to the limit of a sequence of difference quotients of its (continuous) indefinite integral function.

**(9.2) Integrability Theorem.** *If $f\colon I \to \boldsymbol{R}$ is measurable on $I$ and if there exist $g, h \in \mathscr{R}^*(I)$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in I$, then $f \in \mathscr{R}^*(I)$.*

The proof uses the fact that $f$ is the limit almost everywhere of a sequence of simple functions and the Dominated Convergence Theorem (8.2).

We say that a set $A \subseteq I = [a,b]$ is *measurable* if its characteristic function is a measurable function (or, equivalently, belongs to $\mathscr{R}^*(I)$). One can show that the sets

$$A \cap B, \quad A \cup B, \quad \text{and} \quad I - A$$

are measurable sets in $I$ whenever $A, B$ are measurable. Thus the collection $\mathscr{M}(I)$ of all measurable subsets of $I$ is an algebra of sets, and the Monotone Convergence Theorem (8.1) implies that $\mathscr{M}(I)$ is a $\sigma$-algebra of sets. Since $\mathscr{M}(I)$ contains all intervals in $I$, it follows that it contains the *Borel measurable* subsets of $I$. Since $\mathscr{M}(I)$ contains all null subsets of $I$, it follows that it is precisely the collection of all *Lebesgue measurable* subsets of $I$.

**§10. FINAL COMMENTS.** It is easy to see that everything extends to complex-valued functions, or to functions with values in $\boldsymbol{R}^m$, $m > 1$.

The theory can also be extended to functions whose domain is a non-compact interval by using a simple device that is discussed in the books of McLeod and of DePree and Swartz.

The main outlines of the theory carry over easily for functions defined on a compact rectangle in $\boldsymbol{R}^m$, $m > 1$; see the books of Mawhin, McLeod, and Pfeffer that are cited below. There are certain complications when the domain is not compact, but the major parts of the theory extend. One of the active areas of research in this topic is in adapting the integral so that a version of the Divergence Theorem with minimal hypotheses holds. The reader is referred to the book of Pfeffer for an account of this work, and to papers of Jarník, Jurkat, Kurzweil, Mawhin, Nonnenmacher, Pfeffer and others for more detail. Some very significant results have been obtained in this direction, but it seems fair to say that a completely satisfactory theory has not yet been established.

In the preceding discussion the domains of the functions have been assumed to belong to one of the spaces $\boldsymbol{R}^m$. Some important steps have been taken to extend the theory to more general domains; we refer the reader to the recent book of Henstock for more details and a very comprehensive bibliography.

There is an account of the history of this material in the books of McLeod and Henstock. The most complete account of the theory in $\mathscr{R}^*([a, b])$ is in the excellent recent book by Gordon, where it is proved that the generalized Riemann integral (there called the Henstock integral) is equivalent to the integrals of Denjoy and Perron.

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